

The Spectrum of Generalized Petersen Graphs

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Abstract

In this paper, we completely describe the spectrum of the generalized Petersen graph $P(n, k)$, thus adding to the classes of graphs whose spectrum is completely known.

1 Introduction and motivation

Let $G = (V(G), E(G))$ be a simple graph. The spectrum of a graph G is the multiset of eigenvalues of the adjacency matrix. The graph spectrum is an important tool one can use to find information about the physical properties of a network, such as robustness, diameter, connectivity [3]. In this research we completely describe the spectrum for the class of graphs, defined below.

The *generalized Petersen graph* (GPG) $P(n, k)$ has vertices, respectively, edges given by

$$\begin{aligned} V(P(n, k)) &= \{a_i, b_i, 0 \leq i \leq n-1\}, \\ E(P(n, k)) &= \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} \mid 0 \leq i \leq n-1\}, \end{aligned}$$

where the subscripts are expressed as integers modulo n ($n \geq 5$), and k is the “skip”. Note that $k \leq \lfloor \frac{n-1}{2} \rfloor$, because of the obvious isomorphism $P(n, k) \cong P(n, n-k)$. Let $A(n, k)$ (respectively, $B(n, k)$) be the subgraph of $P(n, k)$ consisting of the vertices $\{a_i \mid 0 \leq i \leq n-1\}$ (respectively, $\{b_i \mid 0 \leq i \leq n-1\}$) and edges $\{a_i a_{i+1} \mid 0 \leq i \leq n-1\}$ (respectively, $\{b_i b_{i+k} \mid 0 \leq i \leq n-1\}$). We will call $A(n, k)$ (respectively, $B(n, k)$) the *outer* (respectively, *inner*) subgraph of $P(n, k)$. We display in Figure 1 the graph $P(12, 3)$.

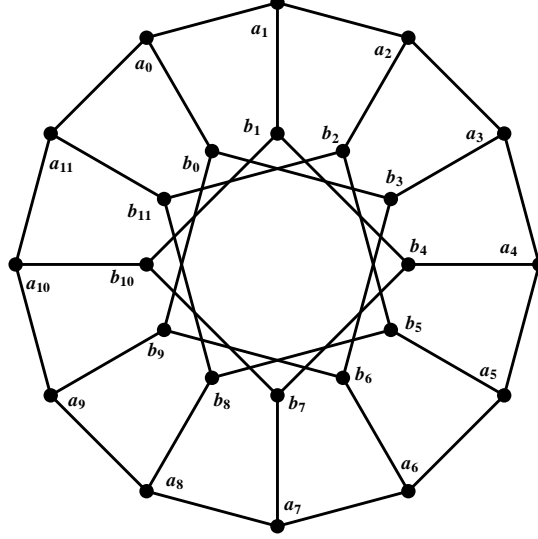
For other graph theoretical terminology the reader could refer to [7].

2 Eigenvalues of $P(n, k)$

In this section we find our description for the spectrum of generalized Petersen graphs $P(n, k)$. We denote the adjacency matrix of the GPG $P(n, k)$

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Figure 1: The Generalized Petersen Graph $P(12, 3)$

by $A(P(n, k))$. Let $\lambda_0 = 3 > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{2n-1}$ be the sequence of eigenvalues of $P(n, k)$.

We call an $n \times n$ matrix *circulant*, and denote it by $\text{circ}(a_1, a_2, \dots, a_n)$ if it is of the form

$$\text{circ}(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ \vdots & & & & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix}.$$

Lemma 2.1 *The $(2n) \times (2n)$ adjacency matrix of the GPG $P(n, k)$ has the block form*

$$A(P(n, k)) = \begin{pmatrix} C_k^n & I_n \\ I_n & C^n \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix, C_k^n, C^n are circulant matrices, with $C_k^n = \text{circ}(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$ and $C^n = \text{circ}(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$ being the adjacency matrix for $A(n, k)$ and $B(n, k)$, respectively. Thus, C^n

is the adjacency matrix of a cycle graph on n vertices \mathcal{C}_n , respectively, C_k^n is the union of d cycle graphs $\mathcal{C}_{n/d}$ on n/d vertices, where $d = \gcd(n, k)$.

Proof. The outer subgraph (whose adjacency matrix is C^n) of $P(n, k)$ is the cycle graph \mathcal{C}_n and the inner subgraph (whose adjacency matrix is C_k^n) has d connected components each isomorphic to $\mathcal{C}_{n/d}$. Also, the adjacency matrix (which depends on the labeling) has the claimed form where the labels used on the outer subgraph are consecutively $1, 2, \dots, n$, and on the inner subgraph the adjacent labels are $i, i + k, i + 2k, \dots$ (where $i + sk$ is understood as $1 + (i - 1 + sk) \pmod{n}$). Note that b_0 is adjacent to vertex b_k in the subgraph and to vertex labeled b_{n-k} in $B(n, k)$, and so $C_k^n = \text{circ}(\underbrace{0, \dots, 0}_{k \text{ times}}, 1, 0, \dots, 0, 1, \underbrace{0, \dots, 0}_{k-1 \text{ times}})$ \square

We recall the Chebyshev's polynomial of the first kind [5], defined by the identity $T_n(\cos \theta) = \cos(n\theta)$, with the generating function $\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-xt}{1-2xt+t^2}$. We now present the eigenvectors and eigenvalues for \mathcal{C}_n (see [2, p. 53 and pp. 72–73]). Let \mathbf{v}^t denote the transpose of \mathbf{v} .

Lemma 2.2 *The eigenvalues of the cycle graph \mathcal{C}_n on n vertices are*

$$\alpha_j = 2 \cos \left(\frac{2\pi j}{n} \right)$$

with a corresponding eigenvector

$$\mathbf{v}_j = (1, \zeta^j, \zeta^{2j}, \dots, \zeta^{(n-1)j})^t,$$

$0 \leq j \leq n-1$. The characteristic polynomial of the cycle \mathcal{C}_n is $2T_n(x/2) - 2$ where T_n is the Chebyshev's polynomial of the first kind.

Corollary 2.3 *The eigenvalues corresponding to the circulant C in the adjacency matrix $A(P(n, k))$ are $\alpha_j = 2 \cos \left(\frac{2\pi j}{n} \right)$ ($0 \leq j \leq n-1$), and the eigenvalues corresponding to C_k are $\beta_j = 2 \cos \left(\frac{2\pi jk}{n} \right)$ ($0 \leq j \leq n-1$).*

We now state our main theorem which adds to the class of graphs whose spectrum is now known.

Theorem 2.4 *The eigenvalues of $P(n, k)$, say $\delta_{2j}, \delta_{2j+1}$, are all roots of the quadratic equation*

$$\delta^2 - (\alpha_j + \beta_j)\delta + \alpha_j\beta_j - 1 = 0, \quad (1)$$

where $\alpha_j = 2 \cos \left(\frac{2\pi j}{n} \right)$, $\beta_j = 2 \cos \left(\frac{2\pi jk}{n} \right) = 2T_k(\alpha_j/2)$ ($0 \leq j \leq n-1$) are the eigenvalues of C , respectively C_k .

Proof. We first consider the case of $d = \gcd(n, k) = 1$. Since $d = 1$, then C_k is the adjacency matrix of a cycle graph isomorphic to \mathcal{C}_n , and so it is similar to C , that is, there exists a permutation matrix P , such that $P^{-1}C_kP = C$. This implies that the two matrices will have the same eigenvalues and eigenvectors. Then α_j, β_j are eigenvalues corresponding to the same eigenvector, say $\mathbf{v}_j = (1, \zeta_n^j, \dots, \zeta_n^{(n-1)j})^t$. We are looking for an eigenvector for $A(P(n, k))$ of the form $\mathbf{w}_j = (a_j \mathbf{v}_j, \mathbf{v}_j)^t$, where a_j will be determined later. If two distinct values for a_j are to be found, for any $0 \leq j \leq n-1$, then we are done with our search for the eigenvectors/eigenvalues.

With this value for \mathbf{w}_j , we need δ (dependent on j) such that

$$\begin{pmatrix} C_k & I_n \\ I_n & C \end{pmatrix} \begin{pmatrix} a_j \mathbf{v}_j \\ \mathbf{v}_j \end{pmatrix} = \delta \begin{pmatrix} a_j \mathbf{v}_j \\ \mathbf{v}_j \end{pmatrix}$$

and so, we get the system

$$\begin{cases} a_j C_k \mathbf{v}_j + \mathbf{v}_j = \delta a_j \mathbf{v}_j \\ a_j \mathbf{v}_j + C \mathbf{v}_j = \delta \mathbf{v}_j \end{cases} \iff \begin{cases} a_j \beta_j \mathbf{v}_j + \mathbf{v}_j = \delta a_j \mathbf{v}_j \\ a_j \mathbf{v}_j + \alpha_j \mathbf{v}_j = \delta \mathbf{v}_j \end{cases}$$

which implies

$$\begin{cases} a_j(\delta - \beta_j) \mathbf{v}_j = \mathbf{v}_j \\ (\delta - \alpha_j) \mathbf{v}_j = a_j \mathbf{v}_j \end{cases}$$

and so, $(\delta - \beta_j)(\delta - \alpha_j) = 1$, which renders the claim for this case, that is, δ must satisfy the equation $\delta^2 - (\alpha_j + \beta_j)\delta + \alpha_j\beta_j - 1 = 0$.

The case of $d > 1$ is treated similarly. The eigenvectors \mathbf{w}_j must have the form $\mathbf{w}_j = (a_1 \mathbf{v}'_j, a_2 \mathbf{v}'_j, \dots, a_d \mathbf{v}'_j, \mathbf{v}_j)$, with \mathbf{v}_j as before and $\mathbf{v}'_j = (1, \zeta_n^j, \dots, \zeta_n^{(n'-1)j})^t$, $n' = n/d$, for some appropriate multipliers a_i . A similar system to the one for $d = 1$ case will be obtained and, interestingly enough, the same polynomial whose roots are the eigenvalues λ_i will be found. The theorem is proved. \square

Using the quadratic formula in (1) and simplifying we get the following corollary.

Corollary 2.5 *The eigenvalues of $P(n, k)$ are given by*

$$\cos\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{2\pi jk}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{2\pi jk}{n}\right)\right)^2 + 1}, \quad 0 \leq j \leq n-1.$$

The largest eigenvalue of $P(n, k)$, $\lambda_0 = 3$, is one of the two values obtained for $j = 0$ in the previous corollary. It is known (see [2, Thm. 3.11]) that if a graph is bipartite, then its spectrum is symmetric with respect to 0. In our case, we have the following result.

Corollary 2.6 *If n is even and k is odd, then the eigenvalues of the bipartite graph $P(n, k)$ are given by ± 3 and*

$$\begin{aligned} & \cos(2j\pi/n) + \cos(2jk\pi/n) \pm \sqrt{(\cos(2j\pi/n) - \cos(2jk\pi/n))^2 + 1} \\ & - \cos(2j\pi/n) + (-1)^k \cos(2jk\pi/n) \mp \sqrt{(\cos(2j\pi/n) + (-1)^k \cos(2jk\pi/n))^2 + 1}, \end{aligned}$$

for $0 \leq j < n/2$.

3 Bounds on the eigenvalues of $P(n, 2)$

In the previous section we found the complete set of eigenvalues of $P(n, k)$ under no restrictions on n and k . Here, we would like to find some bounds on some eigenvalues. Eigenvalue interlacing techniques (see the great survey by Haemers [4] on the topic) will not work easily since there is no visible connection between the various $P(n, k)$, and moreover, the technique is not sensitive enough for our purpose. We shall use a different method.

Here, we will take $k = 2$ and consider $P(n, 2)$ (this includes the case of the classical Petersen graph $P(5, 2)$). Since the second Chebyshev polynomial of the first kind is $T_2(x) = 2x^2 - 1$, we immediately obtain the following:

Theorem 3.1 *The eigenvalues of $P(n, 2)$ are (for $0 \leq j \leq n-1$)*

$$2 \cos^2(2j\pi/n) + \cos(2j\pi/n) - 1 \pm \sqrt{(2 \cos^2(2j\pi/n) - \cos(2j\pi/n) - 1)^2 + 1}.$$

To find good bounds on the eigenvalues in this case, we look for the extreme points of the two functions

$$f_{\pm}(x) = 2x^2 + x - 1 \pm \sqrt{(2x^2 - x - 1)^2 + 1}, \quad (2)$$

in the interval $-1 \leq x \leq 1$. Certainly, we cannot expect exact or even tight results, in general, since the sequence $\frac{2j\pi}{n}$, $0 \leq j < n-1$, is finite and therefore, $\cos\left(\frac{2j\pi}{n}\right)$ is not dense in this interval. However, we will have lower

and upper bounds, which is what we are interested in. Since any differentiable function in a compact domain attains its extreme points at either the critical points or on the boundary, we proceed by studying first the functions' critical points:

$$f'_{\pm}(x) = 4x + 1 \pm \frac{(2x^2 - x - 1)(4x - 1)}{\sqrt{(2x^2 - x - 1)^2 + 1}} = 0,$$

has solutions (computed by Mathematica) at $x_1 \sim -0.41100$ (for f_+) and $x_2 \sim -0.65041$, $x_3 \sim -0.04610$ (for f_-). The values of the corresponding f_{\pm} at these critical points are

$$\begin{aligned} f_+(-0.41100) &= -0.04210\dots \\ f_-(-0.65041) &= -1.92081\dots \\ f_-(-0.04610) &= -2.42092\dots \end{aligned}$$

Further, we look at the values of f_{\pm} at $|x| = 1$. Thus, $f_+(1) = 3$, $f_+(-1) = \sqrt{5}$, and $f_-(1) = 1$, $f_-(-1) = -\sqrt{5}$. Certainly, the maximum value is 3, and the minimum value is approximately -2.42092. We sketch in Figure 2 the two functions f_{\pm} , to visualize our analysis from above:

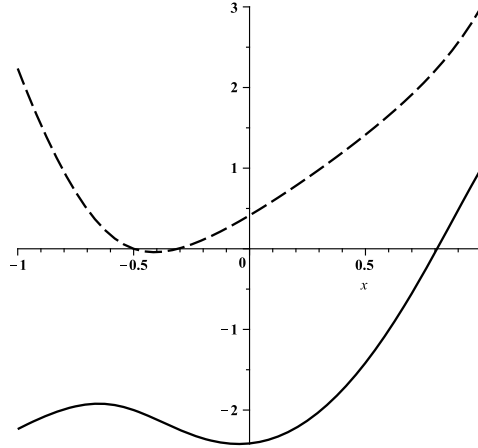


Figure 2: The top function is f_+ and the bottom function is f_-

Every value of f_+ is above every value of f_- , and so the minimum is attained by f_- and the maximum is attained by f_+ . Furthermore, we see

that the second largest eigenvalue of $P(n, 2)$ is

$$\begin{aligned}\lambda_1 &= f_+ \left(\cos \left(\frac{2\pi}{n} \right) \right) \\ &= \cos \left(\frac{2\pi}{n} \right) + \cos \left(\frac{4\pi}{n} \right) + \sqrt{4 \left(2 \cos \left(\frac{2\pi}{n} \right) + 1 \right)^2 \sin^4 \left(\frac{\pi}{n} \right) + 1},\end{aligned}\tag{3}$$

which increases as n increases (shown simply by using Calculus techniques). For instance, for $3 \leq n \leq 20$ the sequence $\lambda_1 = \lambda_1(n)$ is

$$0, 0.41421, 1., 1.41421, 1.71083, 1.93185, 2.10199, 2.23607, 2.34356, 2.43091, \\ 2.50268, 2.56224, 2.61211, 2.65421, 2.69002, 2.7207, 2.74716, 2.77011.$$

Since $\lim_{n \rightarrow \infty} \cos \left(\frac{2\pi}{n} \right) = 1$, we obtain the next result.

Theorem 3.2 *The eigenvalues of $P(n, 2)$ are*

$$\lambda_0 = 3 > \lambda_1 \geq \dots \geq \lambda_{2n-1} \geq -2.42092.$$

Moreover, the second largest eigenvalue satisfies $\lim_{n \rightarrow \infty} \lambda_1(n) = 3$.

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4 Further comments

All of our results for $P(n, 2)$ can be certainly extended to $P(n, 3)$, $P(n, 4)$, etc., but to find sensitive bounds on eigenvalues for arbitrary GPG $P(n, k)$ does not seem to be easy, since the sequence of the involved Chebyshev's polynomials of the first kind does not have a “controllable” behavior in $|x| \leq 1$.

Also, it would be interesting to investigate the number of and distinct values among the eigenvalues of $P(n, k)$, and that is presumably doable. We suspect that the methods of this paper can be also applied to the I -graphs of [1] or the *supergeneralized* Petersen graphs of [6].

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